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ORTHOGONAL EXPANSIONS FOR GENERALIZED GEGENBAUER WEIGHT FUNCTION ON THE UNIT BALL

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ABSTRACT. Orthogonal polynomials and expansions are studied for the weight function $h_\kappa^2(x)\|x\|^{2\nu}(1-\|x\|^2)^{\mu-1/2}$ on the unit ball of $\mathbb{R}^d$, where h_κ is a reflection invariant function, and for related weight function on the simplex of $\mathbb{R}^d$. A concise formula for the reproducing kernels of orthogonal subspaces is derived and used to study summability of the Fourier orthogonal expansions.

1. INTRODUCTION

Fourier orthogonal expansions on the unit ball $\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| \le 1\}$ of $\mathbb{R}^d$ have been studied intensively in recent years ([3, 5]) for the classical weight function

$$W_\mu(x) := (1-\|x\|^2)^{\mu-1/2}, \qquad \mu > -1/2,$$

and, more generally, for the weight functions

$$W_{\kappa,\mu}(x) := h_\kappa^2(x)(1-\|x\|^2)^{\mu-1/2}, \qquad \mu > -1/2,$$

where h_κ is certain weight function that is invariant under a reflection group. Much of the progress is based on our understanding of orthogonal structure, encapsulated in the concise formulas for the reproducing kernels of orthogonal spaces that are integral kernels of orthogonal projection operators. These concise formulas serve as an essential tool for studying orthogonal expansions and allow us to define a meaningful convolution structure on the unit ball. As an example, let $\mathcal{V}_n^d(W_\mu)$ be the space of orthogonal polynomials of degree n with respect to W_μ on $\mathbb{B}^d$. Then the reproducing kernel $P_n(W_\mu;\cdot,\cdot)$ of this space satisfies the relation ([7])

$$P_n(W_\mu;x,y) = c_\mu \int_{-1}^{1} Z_n^{\mu+\frac{d-1}{2}}\left(\langle x,y\rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\,t\right)(1-t^2)^{\mu-1}dt, \tag{1.1}$$

where $x, y \in \mathbb{B}^d$, c_μ is a the normalization constant so that the integral is 1 when $n = 0$ and Z_n^λ is a multiple of the Gegenbauer polynomial C_n^λ, defined by

$$Z_n^\lambda(t) := \frac{n+\lambda}{\lambda} C_n^\lambda(t), \qquad \lambda > 0, \quad -1 \le t \le 1. \tag{1.2}$$

The orthogonal structure on the unit ball is closely related to that on the unit sphere, so much so that the results on the ball can be deduced from the theory of h-harmonics with respect to the reflection group. The concise formula (1.1) plays the role of the reproducing kernel (zonal harmonic) for spherical harmonics.

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In the present paper, we consider the weight function of the form

$$W_{\kappa,\mu,\nu}(x) := h_\kappa^2(x)\|x\|^{2\nu}(1-\|x\|^2)^{\mu-1/2},$$

which we shall call the generalized Gegenbauer weight function on the ball and we shall write $W_{\mu,\nu} := W_{0,\mu,\nu}$ when $h_\kappa(x) \equiv 1$. The additional factor $\|x\|^{2\nu}$, which introduces a singularity at the origin of the unit ball, breaks down the connection to the theory of h-harmonics. Some properties already established for $W_{\kappa,\mu,0}$ do not extend to the setting of $W_{\kappa,\mu,\nu}$; for example, orthogonal polynomials for $W_{\mu,\nu}$ are no longer eigenfunctions of a second order linear differential operators. On the other hand, a basis of orthogonal polynomials can still be deduced in polar coordinates and we can still deduce a concise formula for the reproducing kernel, based on an integral relation for the Gegenbauer polynomials discovered recently in [9]. The latter was derived for $W_{\mu,\nu}$ in [9], it motivates our study here and opens the possibility of carrying out analysis on the ball with respect to the weight function $W_{\kappa,\mu,\nu}$. Our goal in this paper is to explore what is still possible and what might be amiss.

There is a close relation between orthogonal structure on the unit ball and the standard simplex of $\mathbb{R}^d$, which allows us to consider orthogonal polynomials and expansions for the weight functions such as

$$U_{\kappa,\mu,\nu}(x) := \prod_{i=1}^d x_i^{\kappa_i-\frac{1}{2}}|x|^\nu(1-|x|)^{\mu-\frac{1}{2}}, \qquad |x| := x_1+\ldots+x_d,$$

on the simplex $\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge 0, |x| \le 1\}$.

The paper is organized as follows. In the next section we recall necessary definitions and study orthogonal polynomials with respect to $W_{\kappa,\mu,\nu}$ on the ball. The concise formula for the reproducing kernel and orthogonal expansions are studied in the third section. The orthogonal structure and expansion on the simplex is studied in the fourth section.

2. Orthogonal polynomials on the unit ball

We start with the definition of the weight function h_κ. Let G be a finite reflection group with a fixed positive root system R_+. Let σ_v denote the reflection along $v \in R_+$, that is, $x\sigma_v = x - 2\langle x, v\rangle/\|v\|^2$ for $x \in \mathbb{R}^d$, where $\langle\cdot,\cdot\rangle$ denote the usual Euclidean inner product of $\mathbb{R}^d$. Let $\kappa : R_+ \mapsto \mathbb{R}$ be a multiplicity function defined on R_+, which is a G-invariant function, and we assume that $\kappa(v) \ge 0$ for all $v \in R_+$. Then the function

$$h_\kappa(x) = \prod_{v\in R_+} |\langle x, v\rangle|^{\kappa(v)}, \qquad x \in \mathbb{R}^d, \tag{2.1}$$

is a positive homogeneous G-invariant function of order $\gamma_\kappa := \sum_{v\in R_+}\kappa_v$. The simplest case is when $G = \mathbb{Z}_2^d$ for which

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \qquad \kappa_i \ge 0. \tag{2.2}$$

We consider orthogonal polynomials for the weight function $W_{\kappa,\nu,\mu}$ on the unit ball

$$W_{\kappa,\mu,\nu}(x) = h_\kappa^2(x)\|x\|^{2\nu}(1-\|x\|^2)^{\mu-1/2}, \quad \mu > -1/2, \quad \nu+\gamma_\kappa+d/2 > 0, \tag{2.3}$$

where h_κ is as in (2.1). It is easy to verify, in polar coordinates, that restrictions on μ and ν guarantee that this weight function is integrable on $\mathbb{B}^d$. We further denote

$W_{\mu,\nu} := W_{0,\mu,\nu}$ and $W_\mu := W_{\mu,0}$. With respect to $W_{\kappa,\mu,\nu}$ we define an inner product

$$\langle f, g\rangle_{\kappa,\mu,\nu} := b_{\kappa,\mu,\nu} \int_{\mathbb{B}^d} f(x)g(x)W_{\kappa,\mu,\nu}(x)dx, \tag{2.4}$$

where $b_{\kappa,\mu,\nu}$ is the normalization constant such that $\langle 1,1\rangle_{\kappa,\mu,\nu} = 1$. Let Π_n^d denote the space of polynomials of degree at most n in d variables. A polynomial $P \in \Pi_n^d$ of degree n is called an orthogonal polynomial with respect to $W_{\kappa,\mu,\nu}$ if $\langle P, Q\rangle_{\kappa,\mu,\nu} = 0$ for all polynomials $Q \in \Pi_{n-1}^d$. Let $\mathcal{V}_n^d(W_{\kappa,\mu,\nu})$ be the space of orthogonal polynomials with respect to the inner product (2.4). Then $\dim \Pi_n^d = \binom{n+d-1}{n}$. A basis $\{P_{j,n}\}$ for $\mathcal{V}_n^d(W_{\kappa,\mu,\nu})$ is called mutually orthogonal if $\langle P_{j,n}, P_{k,n}\rangle_{\kappa,\mu,\nu} = 0$ whenever $j \neq k$ and it is called orthonormal if, in addition, $\langle P_{j,n}, P_{j,n}\rangle_{\kappa,\mu,\nu} = 1$. There are many different bases for the space $\mathcal{V}_n^d(W_{k,\mu,\nu})$. The structure of the weight function suggests a particular mutually orthogonal basis that can be constructed explicitly. To state this basis, we need h-spherical harmonics defined by Dunkl, which generalize ordinary spherical harmonics.

Associated with G and κ, the Dunkl operators, $\mathcal{D}_1, \ldots, \mathcal{D}_d$, are first order difference-differential operators defined by ([4])

$$\mathcal{D}_i f(x) = \partial_i f(x) + \sum_{v\in R_+} \kappa(v) \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} v_i,$$

where $v = (v_1, \ldots, v_d)$ and $x\sigma_v := x - 2\langle x, v\rangle v/\|v\|^2$. This family of operators enjoys a remarkable commutativity, $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$, which leads to the definition of the h-Laplacian defined by $\Delta_h := \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2$. An h-harmonic is a homogeneous polynomial that satisfies $\Delta_h = 0$ and its restriction on the unit sphere $\mathbb{S}^{d-1}$ is called spherical h-harmonics, which becomes ordinary spherical harmonic when $\kappa = 0$. Let $\mathcal{H}_n^d(h_\kappa^2)$ be the space of h-harmonic polynomials of degree n. For $n \neq m$, it is known that

$$\langle Y_n^h, Y_m^h\rangle_\kappa := b_\kappa \int_{\mathbb{S}^{d-1}} Y_n^h(x)Y_m^h(x)h_\kappa^2(x)d\sigma = 0, \qquad Y_n \in \mathcal{H}_n^d(h_\kappa^2), \quad Y_m \in \mathcal{H}_m^d(h_\kappa^2),$$

where $d\sigma$ denotes the surface measure on $\mathbb{S}^{d-1}$ and b_κ is the normalization constant such that $\langle 1,1\rangle_\kappa = 1$. In polar coordinates, the h-Laplacian can be written as

$$\Delta_h = \frac{\partial^2}{\partial r^2} + \frac{2\lambda_\kappa + 1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{h,0}, \qquad \lambda_\kappa := \gamma_k + \frac{d-2}{2}, \tag{2.5}$$

where $r = \|x\|$ and $\Delta_{h,0}$ is the spherical part of the h-Laplacian, which has h-harmonics as eigenfunctions. More precisely, if $Y_n^h \in \mathcal{H}_n^d(h_\kappa^2)$, then

$$\Delta_{h,0} Y_n^h(x) = -n(n + 2\lambda_\kappa)Y_n^h(x). \tag{2.6}$$

In the case of $h_\kappa(x) = 1$, Δ_h becomes the ordinary Laplacian and $\Delta_{h,0}$ becomes the Laplace-Beltrami operator.

The h-harmonics can be used as building blocks of orthogonal polynomials on the unit ball. Let $\sigma_m^d := \dim \mathcal{H}_m^d(h_\kappa^2)$ and let $\{Y_{\ell,m}^h : 1 \le \ell \le \sigma_m^d\}$ be an orthonormal basis of $\mathcal{H}_m^d(h_\kappa^2)$, normalized with respect to $\langle \cdot,\cdot\rangle_\kappa$, and let $P_n^{(\alpha,\beta)}(t)$ denote the usual Jacobi polynomial of degree n. Define

$$P_{j,\ell}^n(x) := P_{j,\ell}^n(W_{\kappa,\mu,\nu}; x) = P_n^{(\mu-\frac{1}{2}, n-2j+\nu+\lambda_\kappa)}(2\|x\|^2 - 1)Y_{\ell,n-2j}^h(x). \tag{2.7}$$

Proposition 2.1. *The set* $\{P_{j,\ell}^n : 1 \le \ell \le \sigma_{n-2j}^d, 0 \le j \le n/2\}$ *is a mutually orthogonal basis of* $\mathcal{V}_n^d(W_{\kappa,\mu,\nu})$ *and the norm of* $P_{j,\ell}^n$ *is given by*

$$\langle P_{j,\ell}^n, P_{j,\ell}^n \rangle_{\kappa,\mu,\nu} = \frac{(\nu+\gamma_\kappa+\frac{d}{2})_{n-j}(\mu+\frac{1}{2})_j(n-j+\nu+\mu+\gamma_\kappa+\frac{d-1}{2})}{j!(\nu+\mu+\gamma_\kappa+\frac{d+1}{2})_{n-j}(n+\nu+\mu+\gamma_\kappa+\frac{d-1}{2})} =: H_j^n,$$

where $(a)_n$ *denotes the Pochhammer symbol,* $(a)_n := a(a+1)\cdots(a+n-1)$.

Proof. In polar coordinates, it is easy to see that

$$\langle f, g \rangle_{\kappa,\mu,\nu} = (b_{\kappa,\mu,\nu}/b_\kappa) \int_0^1 \langle f(r\cdot), g(r\cdot) \rangle_\kappa r^{d-1+2\gamma_\kappa+2\nu}(1-r^2)^{\mu-1/2} dr,$$

from which the orthogonality of $P_{j,\ell}^n$ follows from the orthogonality of h-spherical harmonics and of the Jacobi polynomials. The proof is similar to that of classical orthogonal polynomials for W_μ on the unit ball, the details can be worked out as in [5, Prop. 5.2.1]. □

In the case of $\nu = 0$, the orthogonal polynomials are closely related to the h-spherical harmonics associated with $h_\kappa^2(x)|x_{d+1}|^{2\mu}$ on the sphere $\mathbb{S}^d$, so much so that it can be deduced from (2.6) that the orthogonal polynomials in $\mathcal{V}_n^d(W_{\kappa,\mu,0}, \mathbb{B}^d)$ are eigenfunctions of a second order differential-difference equation; more precisely,

$$D_{\kappa,\mu} P = -\eta_n^{\kappa,\mu} P, \qquad \forall P \in \mathcal{V}_n^d(W_{\kappa,\mu,0}, \mathbb{B}^d), \tag{2.8}$$

where $\eta_n^{\kappa,\mu} := n(n+2\lambda_k+2\mu+1)$ and

$$D_{\kappa,\mu} := \Delta_h - \langle x, \nabla \rangle^2 - (2\lambda_\kappa + 2\mu + 1)\langle x, \nabla \rangle.$$

This property plays an important role in the study of Fourier orthogonal expansions on the unit ball; for example, it allows us to define an analogue of the heat kernel operator. One naturally asks if there is an extension of this property for the case $\nu \ne 0$.

For this purpose, it is easier to rewrite the basis in (2.7) in terms of the generalized Gegenbauer polynomials $C_n^{(a,b)}$, which are orthogonal polynomials with respect to the weight function $|t|^b(1-t^2)^{a-1/2}$ on $[-1,1]$ (see [5, Section 1.5]). These polynomials satisfy a difference-differential equation that we record below.

Proposition 2.2. *The Generalized Gegenbauer polynomials* $C_n^{(a,b)}$ *satisfy the equation*

$$(1-t)^2 y''(t) - (2a+2b+1)ty'(t) + \frac{2b}{t}\left(y'(t) - \frac{y(t)-y(-t)}{2t}\right) + n(n+2a+2b)y(t) = 0.$$

In polar coordinates $(x_1, x_2) = r(\cos\theta, \sin\theta)$, the polynomials $r^n C_n^{\kappa_2,\kappa_1}(\cos\theta)$ are h-spherical harmonics associated with $|x_1|^{\kappa_1}|x_2|^{\kappa_2}$ on $\mathbb{S}^1$, so that the above proposition follows from (2.6). It is known that

$$C_{2n}^{(a,b)}(t) = \frac{(a+b)_n}{(b+\frac{1}{2})_n} P_n^{(a-1/2,b-1/2)}(2t^2-1),$$

which are even functions and for which the differential-difference equation in the proposition simplifies to

$$(1-t^2)y'' - (2a+2b+1)ty' + \frac{2b}{t}y' + n(n+2a+2b)y = 0. \tag{2.9}$$

In terms of $C_{2n}^{(\alpha,b)}$, the basis (2.7) becomes

$$P_{\ell,j}^n(W_{\kappa,\mu,\nu}; x) = c(j) C_{2j}^{(\mu, n-2j+\lambda_k+\nu+\frac{1}{2})}(\|x\|) Y_{\ell,n-2j}^h(x),$$

where $c(j)$ is a constant. The differential-difference equation can be verified using the following lemma.

Lemma 2.3. *Let $g(x) = p(\|x\|)Y_{n-2j}^h$ with p being a polynomial of one variable and $Y_{n-2j}^h \in \mathcal{H}_{n-2j}$. In the polar coordinates $x = r\xi$, $\xi \in \mathbb{S}^{d-1}$ and $r \ge 0$,*

$$\begin{aligned}
\Delta_h g(x) &= \left[p''(r) + \frac{2(n-2j)+2\lambda_\kappa+1}{r}p'(r)\right] Y_{n-2j}^h(x), \\
\frac{d}{dr}g(x) &= \left[p'(r) + \frac{n-2j}{r}p(r)\right] Y_{n-2j}^h(x), \\
\frac{d^2}{dr^2}g(x) &= \left[p''(r) + \frac{2(n-2j)}{r}p'(r) + \frac{(n-2j)(n-2j-1)}{r^2}p(r)\right] Y_{n-2j}^h(x).
\end{aligned}$$

Proof. Using the fact that $Y_{n-2j}^h(x) = r^{n-2j}Y_{n-2j}^h(\xi)$, the proof of the first item follows from (2.5) and (2.6). The detail, and the proof of the other two identities, amounts to a straightforward computation. □

In the polar coordinates $x = r\xi$, it is easy to verify that $\langle x, \nabla\rangle = r\frac{d}{dr}$. Hence, using the identities in the lemma, we can give a direct proof of (2.8) as follows: setting $p(r) = C_{2j}^{(\mu, n-2j+\lambda_\kappa+\frac{1}{2})}(r)$ and using (2.9), it is straightforward to verify that (2.8) holds for $P_{\ell,j}^n(W_{\kappa,\mu,\nu}; x)$, which establishes the identity for all elements in $\mathcal{V}_n^d(W_{\kappa,\mu,\nu})$ since the terms in (2.8) are independent of j.

For the case $\nu \ne 0$, we need to apply the lemma with $p_j(r) = C_{2j}^{(\mu, n-2j+\lambda_\kappa+\nu+\frac{1}{2})}(r)$. The same consideration, however, yields the following weaker result:

Proposition 2.4. *The polynomial $P(x) = P_{\ell,j}^n(W_{\kappa,\mu,\nu}; x)$ in (2.7) satisfies*

$$\begin{aligned}
&(\Delta_h - \langle x, \nabla\rangle^2 - (2\lambda_\kappa + 2\mu + 2\nu + 1)\langle x, \nabla\rangle)P \\
&\quad + \frac{2\nu}{\|x\|^2}(\langle x, \nabla\rangle - (n-2j))P = -n(n + 2\lambda_k + 2\mu + 2\nu + 1)P.
\end{aligned} \tag{2.10}$$

The last term in the left hand side in (2.10), which can be written as $p'(\|x\|)Y_{n-2j}^h(x)$ with $p(r) = P_j^{(\mu-1/2, n-2j+\lambda_k+\nu)}(2r^2-1)$ by the second identity in Lemma 2.3, depends on the index j in $P_{\ell,j}^n(W_{\kappa,\mu,\nu}; x)$. This means that the (2.10) works only for $P_{\ell,j}^n(W_{\kappa,\mu,\nu}; x)$ but does not work for all elements in $\mathcal{V}_n^d(W_{\kappa,\mu,\nu})$. This is unfortunate, since the fact that $\mathcal{V}_n^d(W_{\kappa,\mu,0})$ satisfies the equation (2.8) is essential for defining an analogue of the heat kernel operator and for define a K-functional, both of which play an important role in analysis with respect to the weight function $W_{\kappa,\mu,0}$.

We end this section with two relations between orthogonal polynomials that have different ν index.

Proposition 2.5. *Let $\lambda_{\kappa,\nu,\mu} = \lambda_\kappa + \nu + \mu + \frac{1}{2}$. Then the orthogonal polynomials in* (2.7) *satisfy the relations*

$$\begin{aligned}
&(n + \lambda_\kappa + \nu + \mu + \tfrac{1}{2})P_{\ell,j}^n(W_{\kappa,\mu,\nu}; x) \\
&\quad = (j + \mu - \tfrac{1}{2})P_{\ell,j-1}^{n-1}(W_{\kappa,\mu,\nu+1}; x) + (n - j + \lambda_\kappa + \nu + \mu + \tfrac{1}{2})P_{\ell,j}^n(W_{\kappa,\mu,\nu+1}; x),
\end{aligned}$$

and

$$\begin{aligned}
&(n + \lambda_\kappa + \nu + \mu + \tfrac{3}{2})\|x\|^2 P_{\ell,j}^n(W_{\kappa,\mu,\nu+1}; x) \\
&\quad = (j+1)P_{\ell,j+1}^{n+2}(W_{\kappa,\mu,\nu}; x) + (2n - 2j + \lambda_\kappa + \nu + 1)P_{\ell,j}^n(W_{\kappa,\mu,\nu}; x).
\end{aligned}$$

Proof. Using (2.7), these two identities follow from the corresponding identities for the Jacobi polynomials given in [1, (22.7.16)] and [1, (22.7.19)]. □

3. Orthogonal expansions on the unit ball

With respect to the mutually orthogonal basis $\{P_{j,\ell}^n\}$ in the Proposition 2.1, the Fourier coefficient $\widehat{f}_{j,\ell}^n$ of $f \in L^2(W_{\kappa,\mu,\nu}, \mathbb{B}^d)$ is defined by $\widehat{f}_j^n := \langle f, P_{j,\ell}^n \rangle_{\kappa,\mu,\nu}$ and the Fourier orthogonal expansion of f is defined by

$$f = \sum_{n=0}^{\infty} \operatorname{proj}_n^{\kappa,\mu,\nu} f \quad \text{with} \quad \operatorname{proj}_n^{\kappa,\mu,\nu} f(x) := \sum_{0 \le j \le n/2} \sum_{\ell=1}^{\sigma_{n-2j}^d} H_{j,n}^{-1} \widehat{f}_{j,\ell}^n P_{j,\ell}^n(x).$$

The projection operator $\operatorname{proj}_n^{\kappa,\mu,\nu} : L^2(W_{\kappa,\mu,\nu}, \mathbb{B}^d) \mapsto \mathcal{V}_n^d(W_{\kappa,\mu,\nu})$ can be written as

$$\operatorname{proj}_n^{\kappa,\mu,\nu} f(x) = b_{\kappa,\mu,\nu} \int_{\mathbb{B}^d} f(y) P_n(W_{\kappa,\mu,\nu}; x, y) W_{\kappa,\mu,\nu}(y) dy,$$

where $P_n(W_{\kappa,\mu,\nu}; \cdot, \cdot)$ is the reproducing kernel of $\mathcal{V}_n^d(W_{\kappa,\mu,\nu})$ and

$$P_n(W_{\kappa,\mu,\nu}; x, y) := \sum_{0 \le j \le n/2} \sum_{\ell=1}^{\sigma_{n-2j}^d} H_{j,n}^{-1} P_{j,\ell}^n(x) P_{j,\ell}^n(y). \tag{3.1}$$

It is known that the reproducing kernel is independent of the choice of orthonormal bases. For the study of Fourier orthogonal series, it is essential to obtain a concise formula for the reproducing kernel.

First we need a concise formula for the reproducing kernel of the h-spherical harmonics, for which we need the intertwining operator V_κ between the partial derivatives and the Dunkl operators, which is a linear operator uniquely determined by

$$V_\kappa 1 = 1, \qquad V_\kappa \mathcal{P}_n^d = \mathcal{P}_n^d, \qquad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \le i \le d,$$

where $\mathcal{P}_n^d$ is the space of homogeneous polynomials of degree n in d variables. The operator V_κ is known to be nonnegative, but the explicit formula of V_κ is unknown in general. In the case $\mathbb{Z}_2^d$, V_κ is an integral operator given by

$$V_\kappa f(x) = c_\kappa \int_{[-1,1]^d} f(x_1 t, \ldots, x_d t_d) \prod_{i=1}^d (1 + t_i)(1 - t_i)^{\kappa_i - 1} dt, \tag{3.2}$$

where $c_\kappa = \prod_{i=1}^d c_{\kappa_i}$ and $c_a = \Gamma(a + 1/2)/(\sqrt{\pi}\Gamma(a))$ and, if some $\kappa_i = 0$, the formula holds under the limit

$$\lim_{a \to 0+} c_a \int_{-1}^1 f(t)(1 - t^2)^{a-1} dt = \frac{1}{2} \left[f(1) + f(-1) \right]. \tag{3.3}$$

Let $\{Y_{\ell,n}^h : 1 \le \ell \le \sigma_n^d\}$ be an orthonormal basis of $\mathcal{H}_n^d(h_\kappa^2)$. Then the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$ is given by the addition formula of h-spherical harmonics,

$$\sum_{\ell=1}^{\sigma_n^d} Y_{\ell,n}^h(x) Y_{\ell,n}^h(y) = V_\kappa \left[Z_n^{\gamma_\kappa + \frac{d-2}{2}} (\langle \cdot, y \rangle) \right](x), \tag{3.4}$$

where Z_n^λ is a multiple of the Gegenbauer polynomial

$$Z_n^\lambda(t) := \frac{n + \lambda}{\lambda} C_n^\lambda(t), \qquad -1 \le t \le 1.$$

For convenience, we define, for given κ, ν, μ,

$$\lambda_{\kappa,\mu,\nu} := \nu + \mu + \gamma_\kappa + \tfrac{d-1}{2}.$$

Theorem 3.1. *Let $\nu > 0$. If $\mu > 0$,*

$$\begin{aligned}(3.5)\quad P_n(W_{\kappa,\mu,\nu}; x, y) = a_{\kappa,\mu,\nu} \int_{-1}^1 \int_0^1 \int_{-1}^1 & V_\kappa \left[Z_n^{\lambda_{\kappa,\mu,\nu}}(\zeta(\cdot; \|x\|, y, u, v, t)) \right](x') \\ & \times (1-t^2)^{\mu-1} dt u^{\nu-1} (1-u)^{\gamma_k + \frac{d-2}{2}} du (1-v^2)^{\nu - \frac12} dv,\end{aligned}$$

where $a_{\kappa,\mu,\nu}$ is a constant such that the integral is 1 if $n = 0$ and

$$\zeta(\cdot; r, y, u, v, t) := r\, \|y\| uv + r\langle \cdot, y\rangle (1-u) + \sqrt{1-r^2}\sqrt{1-\|y\|^2}\, t;$$

furthermore, if $\mu = 0$, then the formula holds under the limit (3.3).

Proof. By (2.7), (3.1) and the addition formula (3.4),

$$\begin{aligned} P_n(W_{\kappa,\mu,\nu}; x, y) = & \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} H_{j,n}^{-1} P_j^{(\mu - \frac12, \beta_{j,n})}(2\|x\|^2 - 1) P_j^{(\mu - \frac12, \beta_{j,n})}(2\|x\|^2 - 1) \\ & \times \|x\|^{n-2j} \|y\|^{n-2j} V_\kappa \left[Z_{n-2j}^{\gamma_\kappa + \frac{d-2}{2}}(\langle \cdot, y' \rangle) \right](x'), \end{aligned}$$

where $\beta_{j,n} := n - 2j + \lambda_{\kappa,\nu} - \frac12$ and $x = \|x\| x'$. The sum in the right hand side is close to the addition formula for an integral of the Gegenbauer polynomial, except that the index of $Z_{n-2j}^{\lambda_\kappa}$ does not match. This is where the new integration relation on the Gegenbauer polynomials comes in, which states, as shown recently in [9], that

$$Z_n^\lambda(x) = c_\mu \sigma_{\lambda+1,\mu} \int_{-1}^1 \int_0^1 Z_n^{\lambda+\nu}(uv + (1-u)x) u^{\nu-1} (1-u)^\lambda du\, (1-v^2)^{\nu - 1/2} dv,$$

where $\lambda > -1/2$, $\nu > 0$, $\sigma_{\lambda,\mu} := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)\Gamma(\mu)}$ and $c_\mu := \frac{\Gamma(\mu+1)}{\Gamma(\frac12)\Gamma(\mu+\frac12)}$. Using this relation, we can write

$$\begin{aligned} P_n(W_{\kappa,\mu,\nu}; x, y) = c_\mu \sigma_{\lambda+1,\mu} \int_{-1}^1 \int_0^1 V_\kappa \Bigg[& \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} H_{j,n}^{-1} P_j^{(\mu - \frac12, \beta_{j,n})}(2\|x\|^2 - 1) \\ & \times P_j^{(\mu-\frac12, \beta_{j,n})}(2\|x\|^2 - 1) \|x\|^{n-2j} \|y\|^{n-2j} Z_{n-2j}^{\nu + \gamma_\kappa + \frac{d-2}{2}}(s\langle \cdot, y' \rangle + (1-)y) \Bigg](x') \\ & \times s^\nu (1-s)^{\mu-1} ds\, (1-y^2)^{\nu - 1/2} dy. \end{aligned}$$

This gives the stated result since the sum inside the bracket can be summed up as an integral of the Gegenbauer polynomial $Z_n^{\lambda_{\kappa,\mu,\nu}}$. This last step is involved but the detail is similar to the proof of Theorem 3.4 in [9], where the case $\kappa = 0$ is established. $\square$

In the case of $\nu = 0$, a concise formula of the reproducing kernel $P_n(W_{\kappa,\mu,0})$ was established in [8], which can be obtained as the limiting case of (3.5) under the limit process of (3.3). The formula in [8] was deduced from the concise formula for the reproducing kernels of the h-spherical harmonics associated with $h_\kappa^2(x)|x_{d+1}|^{2\mu}$ on the sphere $\mathbb{S}^d$, which are intimately connected to orthogonal polynomials with respect to $W_{\kappa,\mu}$ on $\mathbb{B}^d$. For $\nu \neq 0$, however, this connection no longer holds.

In the case of $G = \mathbb{Z}_2^d$, the intertwining operator V_κ is given explicitly by (3.2). We state this case as a corollary.

Corollary 3.2. *Let $W_{\kappa,\mu,\nu}$ be given in terms of h_κ defined in* (2.2) *and let $\nu > 0$. For $\kappa_i \ge 0$ and $\nu \ge 0$,*

$$P_n(W_{\kappa,\mu,\nu}; x, y) = a_{\kappa,\mu,\nu} \int_{-1}^1 \int_0^1 \int_{-1}^1 \int_{[-1,1]^d} Z_n^{\lambda_{\kappa,\mu,\nu}}(\zeta(x,y,u,v,s,t)) \prod_{i=1}^d (1+s_i)$$
$$\times \prod_{i=1}^d (1-s_i^2)^{\kappa_i - 1} ds (1-t^2)^{\mu-1} dt u^{\nu-1} (1-u)^{\gamma_k + \frac{d-2}{2}} du (1-v^2)^{\nu - \frac{1}{2}} dv,$$

which holds under the limit (3.3) *when μ or any κ_i is 0, where*

$$\zeta(x,y,u,v,s,t) := \|x\| \, \|y\| uv + (1-u) \sum_{i=1}^d x_i y_i s_i + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\, t.$$

According to these concise formulas, $P_n(W_{\kappa,\mu,\nu})$ is an integral transform of the Gegenbauer polynomials, which means that the Fourier orthogonal expansions with respect to $W_{\kappa,\mu,\nu}$ is connected to the orthogonal expansions in the Gegenbauer polynomials. Let $w_\lambda(x) := (1-x^2)^{\lambda - \frac{1}{2}}$ for $\lambda > -1/2$ and $x \in (-1,1)$ and c_λ be the normalization constant of w_λ. The Gegenbauer polynomials C_n^λ are orthogonal with respect to w_λ. For $g \in L^1(w_{\lambda_{\kappa,\mu,\nu}}; [-1,1])$ and $x, y \in \mathbb{B}^d$, define

$$L_x^{\kappa,\mu,\nu} g(y) := a_{\kappa,\mu,\nu} \int_{-1}^1 \int_0^1 \int_{-1}^1 V_\kappa \left[g(\zeta(\cdot; x, y, u, v, t)) \right](x') \tag{3.6}$$
$$\times (1-t^2)^{\mu-1} dt u^{\nu-1} (1-u)^{\gamma_k + \frac{d-2}{2}} du (1-v^2)^{\nu-\frac{1}{2}} dv.$$

For $f \in L^1(W_{\kappa,\mu,\nu}, \mathbb{B}^d)$ and $g \in L^1(w_{\lambda_{\kappa,\mu,\nu}}; [-1,1])$, define

$$(f *_{\kappa,\mu,\nu} g)(x) := b_{\kappa,\mu,\nu} \int_{\mathbb{B}^d} f(y) L_x^{\kappa,\mu,\nu} g(y) W_{\kappa,\mu,\nu}(y) dy.$$

This defines a convolution structure with respect to $W_{\kappa,\mu,\nu}$ on $\mathbb{B}^d$. To develop its property, we start with a lemma.

Lemma 3.3. *Let $\nu \ge 0$ and $\mu \ge 0$, and write $\lambda = \lambda_{\kappa,\mu,\nu}$. Then for $g \in L^1(w_\lambda; [-1,1])$ and $P_n \in \mathcal{V}_n^d(W_{\kappa,\mu,\nu})$,*

$$b_{\kappa,\mu,\nu} \int_{\mathbb{B}^d} L_x^{\kappa,\mu,\nu} g(y) P_n(y) W_{\kappa,\mu,\nu}(y) dy = c_\lambda \int_{-1}^1 \frac{C_n^\lambda(t)}{C_n^\lambda(1)} g(t) w_\lambda(t) dt P_n(x). \tag{3.7}$$

Proof. It follows directly from the definition that

$$P_n(W_{\kappa,\mu,\nu}; x, y) := L_x^{\kappa,\mu,\nu} Z_n^{\kappa,\mu,\nu}(y). \tag{3.8}$$

If g is a polynomial of degree at most m, then g can be written as

$$g(t) = \sum_{k=0}^m \widehat{g}_n^\lambda Z_k^\lambda(t), \quad \text{with} \quad \widehat{g}_n^\lambda := c_\mu \int_{-1}^1 \frac{C_k^\lambda(t)}{C_k^\lambda(1)} g(t) w_\lambda(t) dt, \tag{3.9}$$

where we have used the fact that the L^2 norm of C_n^λ is equal to $C_n^\lambda(1)\lambda/(n+\lambda)$, which implies that

$$L_x^{\kappa,\mu,\nu} g(y) = \sum_{k=0}^n \widehat{g}_n^\lambda P_k(W_{\kappa,\mu,\nu}; x, y), \qquad x, y \in \mathbb{B}^d.$$

Consequently, if $m \ge n$, then by the definition of the reproducing kernel,

$$b_{\kappa,\mu,\nu} \int_{\mathbb{B}^d} L_x^{\kappa,\mu,\nu} g(y) P_n(y) W_{\kappa,\mu,\nu}(y) dy = \widehat{g}_n^\lambda P_n(y),$$

which proves (3.7) for g being a polynomial of degree $m \ge n$ and, hence, for $g \in L^1(w_\lambda; [-1,1])$ by the density of polynomials. $\square$

Proposition 3.4. *Let $\nu \ge 0$ and $\mu \ge 0$. Let $p, q, r \ge 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$. For $f \in L^q(W_{\kappa,\mu,\nu}, \mathbb{B}^d)$ and $g \in L^r(w_{\lambda_{\kappa,\mu,\nu}}; [-1,1])$,*

$$\|f *_{\kappa,\mu,\nu} g\|_{W_{\kappa,\mu,\nu},p} \le \|f\|_{W_{\kappa,\mu,\nu},q} \|g\|_{w_{\lambda_{\kappa,\mu,\nu}},r}. \tag{3.10}$$

Proof. Following the standard proof of Young's inequality, it is sufficient to show that $\|L_x^{\kappa,\mu,\nu} g\|_{W_{\kappa,\mu,\nu},r} \le \|g\|_{w_{\kappa,\mu,\nu},r}$ for $1 \le r \le \infty$. Since V_κ is nonnegative, $|V_\kappa g| \le V_k(|g|)$, it follow that $|L_x^{\kappa,\mu,\nu} g| \le L_x^{\kappa,\mu,\nu}(|g|)$. Hence, the inequality (3.6) holds for $p = \infty$ directly by the definition and for $p = 1$ by applying (3.7). The log-convexity of the L^r-norm establishes the case for $1 < r < \infty$. $\square$

Proposition 3.5. *Let $\nu, \mu \ge 0$ and let $\widehat{g}_n^\lambda$ be the Fourier coefficient of g defined in (3.9). Then for $f \in L^1(W_{\kappa,\mu,\nu}, \mathbb{B}^d)$ and $g \in L^1(w_{\lambda_{\kappa,\mu,\nu}}; [-1,1])$,*

$$\operatorname{proj}_n^{\kappa,\mu,\nu}(f *_{\kappa,\mu,\nu} g)(x) = \widehat{g}_n^{\lambda_{\kappa,\mu,\nu}} \operatorname{proj}_n^{\kappa,\mu,\nu} f(x)$$

This proposition justifies calling $*_{\kappa,\mu,\nu}$ a convolution. Its proof follows easily from (3.7) and from exchange of integrals.

For $\delta > 0$, the Cesàro (C,δ) means $S_n^\delta(W_{\kappa,\mu,\nu}; f)$ of the Fourier orthogonal expansion is defined by

$$S_n^\delta(W_{\kappa,\mu,\nu}; f) := \frac{1}{\binom{n+\delta}{d}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} \operatorname{proj}_k^{\kappa,\mu,\nu} f,$$

which can be written as an integral of f against the kernel $K_n^\delta(W_{\kappa,\mu,\nu}; x, y)$. Let $k_n^\delta(w_\lambda; s, t)$ be the Cesàro (C,δ) means of the Gegenbauer series; then

$$k_n^\delta(w_\lambda; s, 1) = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} Z_k^\lambda(s).$$

As a consequence of (3.8), we can write

$$K_n^\delta(W_{\kappa,\mu,\nu}; x, y) = L_x \left[k_n^\delta(w_{\lambda_{\kappa,\mu,\nu}}; \cdot, 1)\right](y). \tag{3.11}$$

Theorem 3.6. *For $\mu, \nu \ge 0$, the Cesàro (C,δ) means for $W_{\kappa,\mu,\nu}$ satisfy*

1. *if $\delta \ge 2\lambda_{\kappa,\nu,\mu} + 1$, then $S_n^\delta(W_{\kappa,\mu,\nu}; f) \ge 0$ if $f(x) \ge 0$;*
2. *$S_n^\delta(W_{\kappa,\mu,\nu}; f)$ converge to f in $L^1(W_{\kappa,\mu,\nu}; \mathbb{B}^d)$ norm or $C(\mathbb{B}^d)$ norm if $\delta > \lambda_{\kappa,\nu,\mu}$.*

Proof. The first assertion follows immediately from the non-negativity of the Gegenbauer series [6]. For the second one, it is sufficient to show that

$$\max_{x \in \mathbb{B}^d} \int_{\mathbb{B}^d} |K_n^\delta(W_{\kappa,\mu,\nu}; x, y)| W_{\kappa,\mu,\nu}(y) dy$$

is bounded, which can be deduced easily from the fact that the integral of $|k_n^\delta(w_\lambda; t, 1)|$ against w_λ is bounded if $\delta > \lambda$ by using (3.11) and applying (3.7) with $P_n(y) = 1$. $\square$

In the case of $\kappa = 0$, it is shown in [9] that $\delta > \nu + \mu + \frac{d-1}{2}$ is also necessary for the second item in the above theorem. However, for $\nu = 0$, the necessary and sufficient condition is known in the case of $G = \mathbb{Z}_2^d$ as $\delta > \sigma_{\kappa,\mu} := \gamma_\kappa - \min_{1 \le i \le d} \kappa_i + \mu + \frac{d-1}{2}$ ([2]), which requires delicate estimate of the (C,δ) kernel based on the explicit formula in Corollary 3.2. We expect that the necessary and sufficient condition for the second item of the theorem is $\delta > \nu + \sigma_{\kappa,\mu}$.

We can also define the Poisson integral for $f \in L^1(W_{\kappa,\mu,\nu}, \mathbb{B}^d)$ by

$$P_r(W_{\kappa,\mu,\nu}; f) := f *_{\kappa,\mu,\nu} P_r^{\kappa,\mu,\nu},$$

where $0 < r < 1$ and the kernel $P_r^{\kappa,\mu,\nu}$ is defined by

$$P_r^{\kappa,\mu,\nu}(x,y) := L_x^{\kappa,\mu,\nu} P_r, \qquad P_r(t) = \frac{1-r^2}{(1-2rt+r^2)^{\lambda_{\kappa,\mu,\nu}+1}}.$$

The Poisson kernel is non–negative and it satisifies

$$P_r^{\kappa,\mu,\nu}(x,y) = \sum_{n=0}^{\infty} P_n(W_{\kappa,\mu,\nu}; x, y) r^n, \qquad 0 < r < 1.$$

The standard proof for the Poisson integral of orthogonal expansions leads to:

Theorem 3.7. *For $f \in L^p(W_{\kappa,\mu,\nu}, \mathbb{B}^d)$ if $1 \le p < \infty$, or $f \in C(\mathbb{B}^d)$ if $p = \infty$, $\lim_{r\to 1-} \|P_r(W_{\kappa,\mu,\nu}; f) - f\|_{W_{\kappa,\mu,\nu},p} = 0$.*

4. Orthogonal polynomials and expansions on the simplex

There is a close relation between orthogonal polynomials on the unit ball $\mathbb{B}^d$ and those on the simplex

$$\mathbb{T}^d := \{x \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge 0, 1 - |x| \ge 0\}, \quad |x| := x_1 + \ldots + x_d,$$

under the mapping $\psi : x \in \mathbb{B}^d \mapsto (x_1^2, \ldots, x_d^2) \in \mathbb{T}^d$. Assume that h_κ is the reflection invariant weight function in (2.1) that is also invariant under $\mathbb{Z}_2^d$, which means that the reflection group G is a semi-product of a reflection group G_0 and $\mathbb{Z}_2^d$. Associated to this weight function, we define a weight function $U_{\kappa,\mu,\nu}$ on the simplex $\mathbb{T}^d$ by

$$U_{\kappa,\mu,\nu}(x) = h_\kappa(\sqrt{x_1}, \ldots, \sqrt{x_d}) |x|^\nu (1-|x|)^{\mu-1/2}, \quad \nu + \gamma_\kappa + d/2 > 0,\ \mu > -1/2, \tag{4.1}$$

which means that $W_{\kappa,\mu,\nu}(x) = (U_{\kappa,\mu,\nu} \circ \psi)(x) |x_1 \cdots x_d|$, where $W_{\kappa,\mu,\nu}$ is the weight function in (2.3) on $\mathbb{B}^d$. In the case of h_κ in (2.2) associated to the group $\mathbb{Z}_2^d$, the weight function is

$$U_{\kappa,\mu,\nu}(x) = \prod_{i=1}^{d} x_i^{\kappa_i - 1/2} |x|^\nu (1-|x|)^{\mu-1/2}, \quad \kappa_i \ge 0, \tag{4.2}$$

which is the classical Jacobi weight function when $\nu = 0$. The case $\nu \ne 0$ has not been considered up to now.

With respect to $U_{\kappa,\mu,\nu}$ we define the inner product on T^d by

$$\langle f, g\rangle_{\kappa,\mu,\nu}^T := b_{\kappa,\nu,\mu} \int_{\mathbb{T}^d} f(x) g(x) U_{\kappa,\mu,\nu}(x) dx.$$

Let $\mathcal{V}_n^d(U_{\kappa,\mu,\nu}, \mathbb{T}^d)$ be the space of orthogonal polynomials with respect to this inner product. It can be shown, as in the case of $\nu = 0$ (cf. [5, Sect. 4.4]), that $\langle f, g\rangle_{\kappa,\mu,\nu}^T = \langle f\circ\psi, g\circ\psi\rangle_{\kappa,\mu,\nu}$, where $\langle f, g\rangle_{\kappa,\mu,\nu}$ is the inner product on $\mathbb{B}^d$ defined in (2.4) and, as a consequence, ψ induces a one-to-one correspondence between $P \in \mathcal{V}_n^d(U_{\kappa,\mu,\nu}, \mathbb{T}^d)$ and $P \circ \psi \in G\mathcal{V}_{2n}^d(W_{\kappa,\mu,\nu})$, the subspace of $\mathcal{V}_n^d(W_{\kappa,\mu,\nu})$ that contains polynomials invariant under $\mathbb{Z}_2^d$. In particular, let $G\mathcal{H}_m^d(h_\kappa^2)$ be the space that contains h-spherical harmonics in $\mathcal{H}_m^d(h_\kappa^2)$ that are invariant under $\mathbb{Z}_2^d$.

Proposition 4.1. *For $0 \le j \le n$, let $\{Y^h_{\ell,2n-2j} : 1 \le \ell \le \binom{n-j+d-1}{n-j}\}$ be an orthonormal basis of $G\mathcal{H}^d_{2n-2j}(h^2_\kappa)$. Define*

$$P^n_{j,\ell}(U_{\kappa,\mu,\nu};x) := P_n^{(\mu-\frac{1}{2},n-2j+\nu+\lambda_\kappa)}(2|x|^2-1)Y^h_{\ell,2n-2j}(\sqrt{x_1},\ldots,\sqrt{x_d}). \tag{4.3}$$

Then the set $\{P^n_{j,\ell}(U_{\kappa,\mu,\nu}) : 1 \le \ell \le \sigma^d_{n-2j}, 0 \le j \le n\}$ is a mutually orthogonal basis of $\mathcal{V}^d_n(U_{\kappa,\mu,\nu},\mathbb{T}^d)$.

The mapping between orthogonal polynomials on the unit ball and those on the simplex extends to the reproducing kernels for the respective spaces, which allows us to derive a concise formula for the reproducing kernel $P_n(U_{\kappa,\mu,\nu};\cdot,\cdot)$ of $\mathcal{V}^d_n(U_{\kappa,\mu,\nu};\mathbb{T}^d)$, defined similarly as the one on the unit ball, and $P_n(U_{\kappa,\mu,\nu})$ is the kernel function for the projection operator $\operatorname{proj}^{\kappa,\mu,\nu}_{n,\mathbb{T}} : L^2(U_{\kappa,\mu,\nu},\mathbb{T}^d) \mapsto \mathcal{V}^d_n(U_{\kappa,\mu,\nu})$. Indeed, for all $x,y \in \mathbb{T}^d$, it is known ([5, Thm. 4.4.5]) that

$$P_n(U_{\kappa,\mu,\nu};x,y) = 2^{-d}\sum_{\varepsilon\in\mathbb{Z}^d_2} P_{2n}(W_{\kappa,\mu,\nu};\sqrt{x},\varepsilon\sqrt{y}), \tag{4.4}$$

where $\sqrt{x} = (\sqrt{x_1},\ldots,\sqrt{x_d})$ and $\varepsilon u = (\varepsilon_1 u_1,\ldots,\varepsilon_d u_d)$. This identity allows us to deduce a concise formula for $P_n(U_{\kappa,\mu,\nu};x,y)$ from Theorem 3.1, in terms of the Gegenbauer polynomial $Z^{\lambda_{\kappa\nu,\mu}}_{2n}$, which we can rewrite in terms of the Jacobi polynomial $P_n^{(\lambda_{\kappa,\mu,\nu},-1/2)}$ by the quadratic transform between these two polynomials, that is,

$$Z^\lambda_{2n}(t) = \frac{2n+\lambda}{\lambda}C^\lambda_{2n}(t) = p_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(1)p_n^{(\lambda-\frac{1}{2},-\frac{1}{2})}(2t^2-1) =: \Xi^\lambda_n(2t^2-1),$$

where $p^{(a,b)}_n$ denote the orthonormal Jacobi polynomial of degree n. We state this formula explicitly in the case of $G = \mathbb{Z}^d_2$. Recall that $\lambda_{\kappa,\mu,\nu} = \nu+\mu+\gamma_\kappa+\frac{d-1}{2}$.

Theorem 4.2. *Let $W_{\kappa,\mu,\nu}$ be given in terms of h_κ defined in (2.2) and let $\nu > 0$. For $\kappa_i \ge 0$ and $\nu \ge 0$,*

$$\begin{aligned} P_n(U_{\kappa,\mu,\nu};x,y) = a_{\kappa,\mu,\nu}\int_{-1}^1\int_0^1\int_{-1}^1\int_{[-1,1]^d} \Xi_n^{\lambda_{\kappa,\mu,\nu}}(2\zeta(x,y,u,v,s,t)^2-1) \\ \times\prod_{i=1}^d(1-s_i^2)^{\kappa_i-1}ds(1-t^2)^{\mu-1}dtu^{\nu-1}(1-u)^{\gamma_\kappa+\frac{d-2}{2}}du(1-v^2)^{\nu-\frac{1}{2}}dv, \end{aligned}$$

which holds under the limit (3.3) when μ or any κ_i is 0, where

$$\zeta(x,y,u,v,s,t) := \sqrt{|x|}\sqrt{|y|}uv + (1-u)\sum_{i=1}^d\sqrt{x_iy_i}s_i + \sqrt{1-|x|}\sqrt{1-|y|^2}\,t.$$

In the case $\nu = 0$, this formula and its version for more general h_κ are known (cf. [8]); the case $\nu \ne 0$ is new. We can also define a convolution $*^{\mathbb{T}}_{\kappa,\mu,\nu}$ between $f \in L^1(U_{\kappa,\mu,\nu};\mathbb{T}^d)$ and $g \in L^1(w_{\lambda_{\kappa,\mu,\nu}-\frac{1}{2},-\frac{1}{2}},[-1,1])$, where $w_{a,\beta}(t) := (1-t)^a(1+t)^b$. In fact, it can be defined as follows:

$$(f *^{\mathbb{T}}_{\kappa,\mu,\nu} g\circ\psi)(x) := (f\circ\psi) *_{\kappa,\mu,\nu} g(2\{\cdot\}^2-1)(x),$$

where the convolution in the right hand side is the one defined in Section 3. The properties of this convolution can then be deduced from the corresponding results on the unit ball. In particular, Proposition 3.4 holds with the norm of $\|\cdot\|_{U_{\kappa,\mu,\nu},p}$ and $\|\cdot\|_{w_{\lambda_{\kappa,\mu,\nu}-\frac{1}{2},-\frac{1}{2}},p}$. Much of the analysis from this point on can be carried out from the

correspondence between analysis on the ball and on the simplex, just as in the case of $\nu = 0$. We conclude this section with a result on summability.

Let $S_n^\delta(U_{\kappa,\mu,\nu}; f)$ be the Cesàro (C,δ) means of the Fourier orthogonal expansion with respect to $U_{\kappa,\mu,\nu}$ on $\mathbb{T}^d$ and let $K_n^\delta(U_{\kappa,\mu,\nu}; \cdot, \cdot)$ be its kernel, both are defined similarly as the corresponding ones on the unit ball. In particular, we can also write

$$S_n^\delta(U_{\kappa,\mu,\nu}; f) = f *_{\kappa,\mu,\nu}^{\mathbb{T}} k_n^\delta(w_{\lambda_{k,\mu,\nu}-\frac{1}{2},\frac{1}{2}}),$$

where $k_n^\delta(w_{a,b}; s, t) = k_n^\delta(w_{a,b}; s, 1)$ denotes the Cesàro (C,δ) kernel of the Jacobi series for $w_{a,b}$ on $[-1,1]$ with one variable evaluated at 1.

Theorem 4.3. *For $\lambda \ge 0$ and $\mu \ge 0$, the Cesàro (C,δ) means for $U_{\kappa,\lambda,\mu}$ satisfy*

1. *if $\delta \ge 2\lambda_{\kappa,\mu,\nu} + 1$, then $S_n^\delta(U_{\kappa,\mu,\nu}; f) \ge 0$ if $f(x) \ge 0$;*
2. *$S_n^\delta(U_{\kappa,\mu,\nu}; f)$ converge to f in $L^1(U_{\kappa,\mu,\nu}; \mathbb{T}^d)$ norm or $C(\mathbb{T}^d)$ norm if $\delta > \lambda_{\kappa,\nu,\mu}$.*

We can also define the Poisson integral and establish an analogue of Theorem 3.7.

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